Energy Considerations for Hybrid Bus Routes

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To the Editor:

Hybrid buses are incredibly powerful not only because of their environmentally friendly aspects, but also because of the economic advantage they bring. Our team recently conducted an in-depth analysis of the routes and elevation around Ithaca along with TCAT bus routes and found new ways the TCAT bus routes can be optimized. Our analysis of some of the bus routes revealed that using hybrid buses can save hundreds of thousands of dollars. Knowing that the TCAT bus system owns eight hybrid buses, we would like to prioritize in using them. Thus, optimizing the TCAT routes with hybrid buses is a crucial task that will not only give Ithaca a greener environment, but also save it's taxpayer.

There are two parts to our plan; prioritize using hybrid buses on certain routes, and changing one route slightly so that it will be more efficient.

From our analysis, we found out that using one hybrid bus on route 11 saves the TCAT system \$114.38 per bus per day and route 82 saves the TCAT system \$65.41 per bus per day. Since these two routes serve the Ithaca College and the Cornell University community, prioritizing them with hybrid buses also makes sense because of the demands of these densely populated areas. As a result, we recommend the TCAT to send 6 buses to Route 11 to save more than \$179,000 per year, and to send 2 buses to Route 82 to save an additional \$34,000 per year (note that we are not counting weekends). This amounts to more than \$210,000 per year that the TCAT can save.

We also looked for possible improvements to routes and found a minor adjustment to route 81. If the TCAT adopts our route, they will save an additional \$2.915 per day.Though not much more of an improvement, every cent of the taxpayer counts. In fact, saving this over years can reduce the cost significantly. We propose this slight change to route 81: instead of looping from Balch @ Cradit Farm to Appel Commons to Hasbrouck, route 81 should first come to Robert Purcell Community Center to Hasbrouck, then turns around to Robert Purcell Community Center. We attached a picture for clarification:

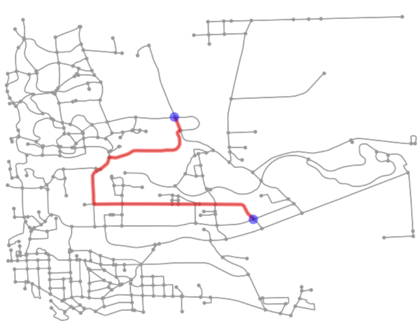

Figure 1: Old Route 81 heading North

In conclusion, we recommend 6 hybrid buses to route 11, 2 hybrid buses to route 82, and a slight change to route 81 as mentioned above. This should save

Figure 2: New Route 81 heading North

hundreds of thousands dollars per year worth of taxpayer's money. We sincerely hope the TCAT consider our model to better our environment and better public transportation in Tompkin's County.

Sincerely,

Dang Pham, Richard Wilde, Zach Whipps

Contents

1 Introduction

The Tompkins Consolidated Area Transport (TCAT) system provides invaluable busing service to all of the members of Tompkins County. There current fleet contains 8 hybrid buses, which use a combination of diesel fuel and electric power from a stored energy source to propel the bus. Additionally, the hybrid buses are equipped with a regenerative braking system, which turns the mechanical stopping energy from the bus into energy that can be stored for use later on.

Because of this, the equations that govern the energy savings by using the buses are highly dependent on what route one takes. For example, it is intuitively obvious that if the bus travels down a steep slope and presses on the brakes the entire way down, it will regenerate more energy than if it were traveling on a flat road.

We want to create a model that will tell us the most effective use of these hybrid buses. That is, we want to limit the use of diesel fuel across all routes, which will minimize the release of pollutants. As an added side benefit, this will save TCAT a significant portion of money because of the reduced use of diesel.

2 Assumptions

- The bus travels at the speed limit on all roads while not accelerating. This applies to gradients as well.
- The bus drivers stop with a reasonable deceleration, such that they never exceed the regenerative breaks' maximum recharge amount.
- Bus drivers will apply the brakes the entire way down any downhill section of road in order to keep the bus at a constant velocity and to regenerate energy.
- The bus has enough diesel fuel to make it throughout the entire route without running out.
- We ignored Route 17, because of logistical nightmares. There are several different buses that are either leaving the garage for the day, or returning from another trip, so it doesn't make sense to check the efficiency. In fact, this is the dispatch route, thus every bus have to leave and return to here. The sum of leaving and returning to this route cancels to zero. Thus, we eliminate Route 17.
- The bus can switch between diesel and electricity instantaneously (prompt)
- the favorable routes in the worst-case scenario are the favorable routes in better scenarios as well

3 Parameters

Physical Parameter	Symbol	Value	Units	Source
Density of Air	ρ_{air}	1.225	kg/m^3	[9]
Rolling Coefficient between Tire and Asphalt	f_f	0.01		[8]
Recharge Efficiency of NiMH Battery	μ_r	.66		[6]
Motor Efficiency	μ_m	.36		[16]
Constant Bus Acceleration	a	0.931	m/s^2	[11]
Mass of Bus	m	18000	kg	[17]
Default Speed Limit	v_s	30	mile per hour	[13]
Drag Coefficient	c	1.28		[?]
Total Air Drag Term	ρ/p	5.096	kg/m	
Downhill $\theta_{critical}$	$\theta_{critical}$	10.5		
Uphill max θ		9.75		

4 Model

4.1 An Energy Based Model

Our approach to this problem was to account for all of the major costs of energy that the bus experiences along its route. This was inspired by a large set of data for Ithaca's roads that we found, which allows us to find the slope, length, and height difference for each road, which correspond very nicely to the factors that we need to know to calculate the gravitational potential energy and energy costs, and from there, all of the necessary physics to model this situation. We will split this up into 4 sections, that will all lead to the final equations that we need to run through our simulation.

4.1.1 Road Cases

We consider two main cases for the road, and each will provide different conditions for how we calculate the energy gained or lost. Our cases are:

1. When you are traveling uphill, and therefore working against Earth's gravity.

2. When the angle, relative to Earth's gravity, is negative, and you are traveling down hill.

In each of these cases, the trip is broken up into 3 sections: an initial accelerating portion to reach the speed specified on the road segment, a constant velocity portion, and a decelerating portion at the end where we slow back to down to zero.

On every road segment we make the assumption that:

1. The bus starts at $v = 0$
2. The bus accelerates at a constant rate to speed v, v varying by the road strip
3. The bus outputs enough energy to combat the dissipative forces and move at a constant velocity, for the part of the road segment between the accelerating and decelerating portions of the strip.
4. The car decelerates at a constant rate
5. The bus ends at $v = 0$
6. All outside forces besides gravity, tire friction, and air friction are negligible.
7. The road is long enough to accelerate and decelerate on without overlap
8. Deceleration and acceleration rates are equal/constant

By observing the graphs in our citation [11], we can approximate the deceleration and acceleration processes by linear equations in time. From there, we averaged the acceleration and deceleration values to get an acceleration of $a = 0.931m/s^2$

The important outside forces are air drag, friction from tires, and gravity; they are governed by

$$F_{tirefriction} = f_r mgcos(\theta), F_g = mgsin(\theta), F_{air} = \rho v^2$$

For case 1, there are a few things we must account for. First, if we are traveling up a hill, then we are fighting gravity, and we must inherently use some extra amount of energy than if we were on a flat surface. The equation for this can be readily found, as we will show.
Extra care must be taken to account for the air friction and the friction due to the rolling of the tires, seeing as they will be significant later on.
During the first leg of an uphill trip, the bus is dragged down by the standard dissipative and inertial forces; in order to accelerate up the hill, our battery will have to apply enough force to counteract these. We therefore expect the battery to output energy; the analytic expression is derived as follows.
Our equations of motion for the bus will be given by:

$$F_{bus} = F_{applied} - mgsin(\theta) - mgf_r cos(\theta) - \rho v^2$$

In order to move at a constant acceleration a, the force applied by the car engine must cancel the dissipative and inertial term; with constant acceleration, we can apply kinematic formulas to obtain:

$$F_{applied} = (ma + mgsin(\theta) + mgf_r cos(\theta)) + 2a\rho x$$

Where x is the distance traveled along the hypotenuse of our angled road. Finally, this implies that our energy equation for the accelerating part of our trip is (where a negative superscript is a loss in energy):

$$\Delta E_1^- = (ma + mgsin(\theta) + mgf_r cos(\theta))v^2/2a + \rho a(v^2/2a)^2$$

The assumption that velocity is constant during the next strip of the road segment is reasonable since we want to keep the trip comfortable for the passengers and abide by standard driving practices. All forces are still against our desired motion, so we again expect a decrease in energy as work will be done by the engine to compensate. The force equation is the same as before, but now with constant v; we integrate with respect to x to find the work done by the battery:

$$\Delta E_2^- = (mgsin(\theta) + mgf_r cos(\theta))(\Delta s - v^2/a)$$

Where Δs is the length of the whole road segment.

For deceleration ,the applied force will be done by the break; since our breaks are regenerative, this will put energy back into the system. By arguments similar to those in the accelerative portion, we obtain:

$$W_{brake} = (-ma + mgsin(\theta) + mgf_r cos(\theta))v^2/2a + \rho(v^2/2a)^2 - a\rho(v^2/2a)^2$$

$$\Delta E_3^+ = \mu_{brake}|W_{brake}|$$

Where

$$0 <= \mu_{brake} < 1$$

is a constant which expresses the efficiency of the regenerative break. The end result is that we can find the energy change for any uphill slope by the equations

$$\Delta E_{loss} = \Delta E_1^- + \Delta E_2^-$$

$$\Delta E_{gain} = \Delta E_3^+$$

For Case 2, gravity will do positive work on our bus. As such, the energy our battery needs to output in order to maintain constant acceleration will decrease with angle. However, given that $a \approx 1$ and that $gsin(\theta)$ can be much greater than one , it is reasonable to assume that there will be an angle for which we no longer apply the battery in order to work with gravity, but the breaks in order to work against gravity.
To derive $\theta_{critical}$ we consider the motion of the bus during the first portion of its trip down the slope. When $\theta < \theta_{critical}$, gravity will work with us, but not enough to create a constant acceleration during the desired leg of the road segment by its self. We must apply some force via the engine in order to compensate. By arguments similar to those in case 1, the work applied via the battery is given by:

$$\Delta E_1 = |((m(a - gsin(\theta)) + (f_r)mgcos(\theta))(v^2/(2a)) + (pa)(v^2/(2a))^2|$$

As θ increases, the necessary work output by the engine will decrease.

At the critical angle where we no longer have to apply force, we assume the term will equate to zero. Solving for theta in this case gives:

$$\theta_{critical} = arcsin((am + pv^2/2)/(mg\sqrt{1 + f_r^2}))$$

If $\theta < \theta_{critical}$, the change in energy associated with this work is negative, since it is supplied by the engine. However, If $\theta > \theta_{critical}$, it will be positive; in this instance the work will be performed by the breaks, and the work performed by the breaks and the engine have equivalent mathematical forms; energy factor is up to a multiple of μ_{brake}.
The derivation of equations in this section follows almost the exact same process as those in the previous section, so we omit their details. Whether θ is less than or greater than $\theta_{critical}$, the rest of the trip will be spent applying the breaks in order to either counter the other forces and maintain speed, or the decelerate to a stop. As such, the change in energy over these section will be positive.
For the strip of constant velocity, we obtain:

$$\Delta E_2^+ = |(mgsin(\theta) - (fmgcos(\theta) + pv^2))((\Delta)s - v^2/a)|$$

and for the strip of deceleration:

$$\Delta E_3^+ = |(m(a - gsin(\theta)) + fmgcos(\theta) + pv^2)(v^2/(2a)) - pa(v^2/2a)^2|$$

Additionally, Because some bus stops are placed within in road segments as opposed to on nodes, more stopping and starting occurs than our above equations account for on these segment. In order to compensate for the extra breaking, we produce another energy term of the form:

$$N * (1 - \mu_{break}) * 1/2mv^2$$

where N is the number of stops on the segment. It corresponds to the energy lost going from zero to v and the energy regained by breaking from v to zero between the stops

4.1.2 Basic Energy Laws on a Road Segment

Since we know several factors about every single road along the bus' route, we start by figuring out exactly how much energy is lost/gained at any possible road situation. Then, we will show how this relates to the ability of the bus to switch between diesel fuel and electric power, and how that will relate to the total amount of diesel versus electric power that is used on every route.

Because the variables that we have are street length, inclination, and change in elevation, we very naturally model each road segment as if it were an inclined plane. Even if the roads were more complexly modeled, the total energy gained or lost over any one road segment would stay the same because of the conservation of energy, without considering dissipative forces. However, we have also decided to include the total air drag and drag due to the rolling of the wheels, meaning that this simplification is necessary.

If ΔE_d is the energy that is lost to diesel over one street segment and ΔE_b is the total energy lost from the battery over a street segment, the the basis of our calculation for the total energy change over one route is

$$E_{used} = \sum_{i=currentroad}^{entireroute} \Delta E_{i,d} + \Delta E_{i,b}$$

Where $\Delta E_{i,b}$ is always 0 if we do not use a hybrid bus. We are essentially just looping through every single subdivision and determining what the energy loss would be.

During each segment of the trip, we take into account the energy gained and lost over while traversing the road segment; we then check to see if we are running on diesel or electricity. If we are running on diesel, the negative changes in energy corresponds to a drop in diesel reserves. If we are running on electricity, they correspond in a drop of charge reserves. The positive changes in energy always correspond to an increase in charge. These changes are dictated by the simple formulas:

$$F_{charge} = (\mu 1) \sum_{segment} -\Delta E^{-} + \Delta E^{+}$$

$$F_{diesel} = (\mu 2) \sum_{segment} -\Delta E^{-}$$

where $\mu 1$ and $\mu 2$ are appropriate conversion factors. Given that we have a maximum and minimal possible charge, we specify that any extra energy attempting to be converted into to a full charge container is wasted, and that if a negative change in energy removes more charge than possible, the remaining change in energy is taken out of the diesel reserves. Additionally, if the electric motor empties, it cannot be used again until the charge goes back over a chosen factor.

It is given that the charge holder has a max rate at which it can regain energy from the regenerative breaks; if the rate at which energy flows into the charger holder from from breaking is less than the max input power, than that power flow is just its self. However, if the rate is greater, than we simply let the flow in be the max input power. This is in accordance we the way most recharging batteries work; that being that the influx rate is almost exactly the flow rate, until acting piece wise and becoming constant after the influx rate becomes greater that the max input power [18]. given the equations for applied forces explained in section 4.1.1, we can calculate the power at which the breaks input energy by using the simple formula $power = v_{bus} * F_{segment}$; this equation is easy to compute given the simple kinematic motion for most segments.

4.2 Limits and Properties of the Bus

4.2.1 The Battery, or Why Hybrid buses Can't Go Uphill

In the problem statement, we saw that these buses can only use diesel when going up slopes that are particularly steep.
In our research, we found that when the bus is being battery operated, there is only a certain maximum working power that the battery can put out at any given time. The power that should be being drawn from the battery is also related to the bus' velocity and angle, so we inferred that the intersection of these two is the maximum velocity that a bus can go uphill.
To find $\theta_{critical}$: Assuming that we're moving up a slope at max constant speed, and that our battery can only exhibit so much power at once, there will exist a theta for which our battery power output maxes out. The battery energy out is $E_b = P\Delta t$, and the dissipative forces take away an energy of

$$\Delta E^{-} = (mgsin(\theta) + mgf_r cos(\theta) + \rho v_f^2)v_f \Delta t$$

setting them equal to one another and solving, to get:

$$P_{max} = mgsin(\theta)v + mgf_r cos(\theta)v_f + \rho v^3$$

This equation can be easily inverted to find theta in terms of v, giving:

$$\theta = arcsin((P_m ax - pv^3)/(vmg\sqrt{1+f_r^2})) - arctan(f)$$

5 Software Implementation

In order to graph, calculate, and optimize TCAT bus routes, we need to find data regarding the roads and data for the TCAT bus stops. There are two main parts we need to accomplish the following in order to achieve our software implementation:

1. Data collection
2. Algorithm implementation

5.1 Data Collection

The model we derived depends heavily on having the following roads information around Tompkins County: one-way streets, speed limits of roads, grade (slope) of the road, and lengths of roads. Of all the resources that we found, there are two notable sources that proved to be invaluable to this project: OpenStreetMap(OSM) and the Google Maps API. As a side note, all of these information can be done with the Google Maps API, however, to get all of them it is required that we purchase premium. Thus, we incorporate data from both OSM and Google Maps to accomplish our task.

OSM gives information on a variety of things, ranging from road information to buildings and many more. Out of these, we found that OSM have extensive information on roads, including their name, speed limit, if the street is one-way or two-way, latitude, and longitude. The one thing we are missing is the elevation, which we then used the Google Maps API to complement our data set.

Figure 3: Route 10 heading north, as seen by our algorithm.

The second part of data collection requires us to find the TCAT bus stops. The only information we need is their latitude and longitude. To do so, we manually found these coordinates by using Google Maps (not the API). The collection of these points proved to be strenuous. However, we attempted to find them at the highest precision possible. Furthermore, we also found the current number of times the bus loops around a route. We present this data below:

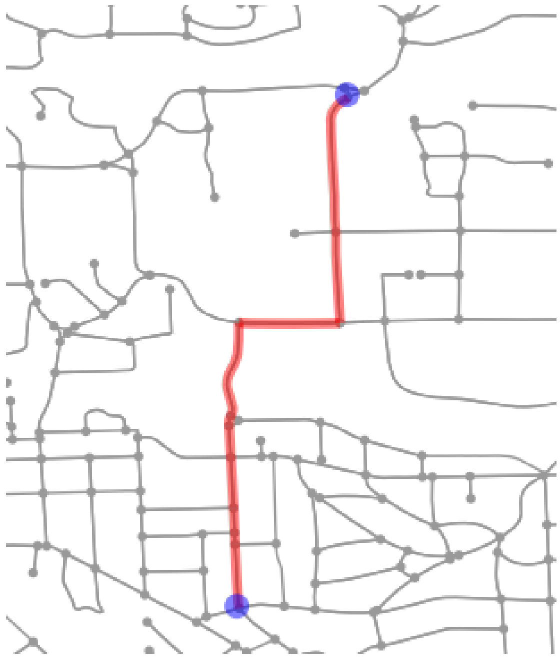

Figure 4: Route 10, heading south. Even though this route is a loop, we had to break it up into parts for the purpose of calculation

Route	Loops per Day
10	78
11	28
15	16
81	15
81	72

5.1.1 Route 15

Route 15 proved to be troublesome for us because it has multiple segments where it passes through parking lots where roads data are not available. We will still attempt to map it the bus stops and run our algorith on this route. However, we will pay special attention and see for any irregularities.

5.2 General Algorithm

The two main goals for our algorithm is to answer the two questions posed:

- Analyze current routes to find the best routes for prioritization
- Find alternative routes (if there any) to optimize efficiency

The general description of what our algorithm implementation is as follow:

1. Get map data of regions around the bus route

2. Create a network of streets (as edges) and intersection (as nodes) using the map data
3. Assign elevation data to nodes by using the Google Maps API
4. Assign the length and speed limit to streets by using data from OSM
5. Calculate the grade (slope) of the street by finding the difference of the elevation between the nodes at the end of the street and dividing that difference by the street length.
6. Assign a weight function to each street. This weight function is an arbitrary function that we defined so that our path finding algorithm avoid very steep slopes. The details of the weight function will be discussed.
7. Now we use the data collected for the bus stops. Snap each of the bus stop to the nearest road intersection. Call this set of nodes corresponding to bus stops an ordered-set A. Call each nodes in A, ζ_i.
8. Apply Dijkstra's Algorithm with the weight as length to find the path that minimize the distance between each ζ_i and ζ_{i+1} until we reach ζ_N where ζ_N is the last bus stop. Summing these paths should result in the same path as a TCAT bus would take.
9. Once we get the route, calculate the amount of energy that the electric battery would use and diesel would use. This will be discussed in detail later.
10. Now apply Dijkstra's Algorithm again with weight as the weight function mentioned in step (6). This should be find a route that has the least slope.
11. Apply (9) on (10). The answer we get from (9) and this step should reveal the efficiency of each route and if there are other optimization to decrease fuel cost.
12. Do this continuously over the whole day to see the total amount of diesel consumption.

5.3 Python and Libraries

We chose to use Python as the main programming language we will use to apply this algorithm. Within Python extensive network of libraries, we found three especially helpful with your endeavors:

- OSMnx: A Python libraries for collecting and make use of street networks by Geoff Boeing [19].
- Networkx: A Python libraries for creating nodes and paths.
- Python Client for Google Maps Services (https://github.com/googlemaps/google-maps-services-python)

OSMnx proved to be extremely helpful as it can handle step 1-4 for us automatically.

5.4 Weight Function

The weight function is an arbitrary function that penalizes on streets with steeper slopes and rewards for shallower slopes and really rewards downhill slopes. To do so, we defined the weight function as:

- If the slope is uphill and steeper than the allowed slope based on the speed limit, then we make the penalty =

$$length \times \sqrt{slope}$$

  since slope is less than 1, so the square root of slope will be large.

- If the slope is uphill and within the allowed slope: penalty =

$$length \times slope^2$$

  since slope is less than 1, so the square of slope will be small.

- If downhill, we favor even more: penalty =

$$5 \times length \times slope^2$$

  since slope is less than 1, so the square of slope will be small. With this weight function, we can now use Dijkstra's Algorithm to find a path that minimizes this weight that accounts for slope. Thus, this will allow to find the least steep route.

5.5 Optimization

The Dijkstra algorithm provided to us by Networkx allows us to find a route between two nodes given some weight function as discussed above. For the first case, we optimize for the length (finding the shortest route between two points). For the second case, we optimize for the slope (finding the route with the least slope between two points). Furthermore, OSMnx allows us to make sure that roads are one way or two way.

5.6 Keeping Track of Energy,Electric Battery, and Diesel Consumption

We can now apply the model over a path segment and calculate the energy lost and gain. This is a trivial implementation and is merely a substitution of formula. Thus, we will refrain from discuss regarding the formulaic implementation further.

However, the most important calculation lies in how much diesel do we use with and without batteries. To do so, we keep track of how much energy we expended in total. This is the the amount of energy that a diesel consumption can supplement. From here we can calculate the amount of diesel consumption without using electric battery. To get the amount of diesel consumption with the electric battery, we need to keep track of how much electric energy is available. Thus for every time we lose energy, we deduct it from the battery and every time we gain energy from brake, we add it into the battery. We assumed that if the battery falls below 20%, then we switch to diesel. When the battery level rises back up to 40%, we then use battery again. Keeping track of these values throughout the calculation will then give us the result we want.

5.7 Validation of Model and Code

From the software implementation we calculated that without using electric battery, the total amount of energy used for a trip of distance 6000 meters around the route of bus 81 is: 9.58×10^7 Joules. Since there is about 35 Mega Joules of energy per liter of Diesel [20], we can calculate that without the electric battery we have used 0.72 gallons worth of diesel. This corresponds to about 5.18 miles per gallon (mpg). A little research around reveals that the mpg for bus is around 3-6 mpg [10]. Our model correctly predicts the amount of diesel used for bus!

6 Results

The results we received from our code is actually not very surprising: most of the TCAT routes we were given were pretty much optimized, and in most of these routes, the electric battery helps fuel consumption significant (ranging from 23% - 44% reduction in diesel consumption). We present our data below:

Route	Optimization	Distance per loop (m)	Diesel energy usage with battery per day (J)	Diesel energy usage without battery per day (J)
10	Length	5043.637308	4599321437	7213179269
10	Slope	6266.630271	5739922870	8728134358
11	Length	8547.746644	2714368450	4119833011
11	Slope	15617.62818	4577261066	7012628433
15	Length	5178.371238	243337947.6	906011704.7
15	Slope	6955.761139	383980654.6	1205490073
81	Length	5965.43154	802363972.1	1436424346
81	Slope	6362.292666	687492989.8	1381831295
82	Length	11612.46741	9573073711	12547021575
82	Slope	14032.65769	11066694103	14717832325

This reveals some answer to if we want to make minor route changes. In most of these cases, it can seen that the TCAT route is the most optimized route. Except for in the case of route 81 where we propose a new route: we remove the bus stops at Appel Commons and at Balch/Cradit Farm and make the bus loops back through Jessup Rd. This route will travel an additional 400 meters, however, will be saving 1 gallons of diesel (following a derivation like the one in previous section on model verification).The old and new route is shown below (figures 5 to 8):

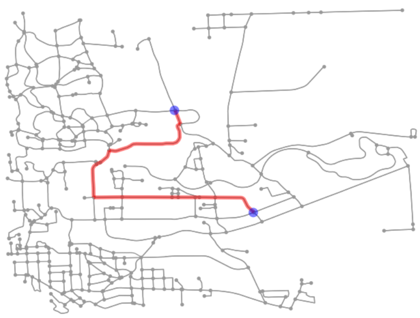

Figure 5: Old Route 81 heading North

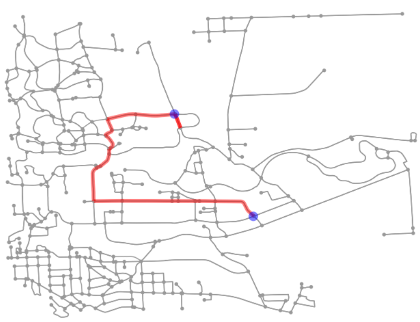

Figure 6: New Route 81 heading North

For the case of length optimization and following the steps similar to that in the model justification in the previous section, the above table corresponds to:

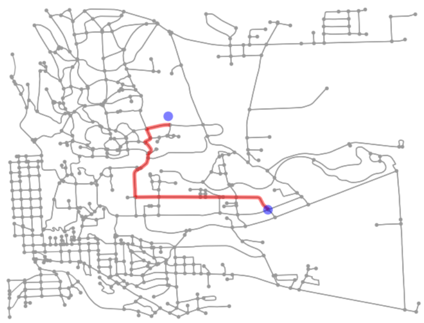

Figure 7: Old Route 81 heading South

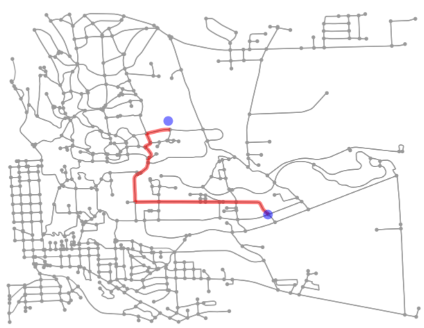

Figure 8: New Route 81 heading South

Route	Diesel used with battery per day(gallons)	Diesel used without battery per day (gallons)	Fuel Saved with battery (gallons)	Fraction of energy saved by using battery
81	6.056	10.84	4.784	44.1%
10	34.71	54.44	19.73	36.2%
15	1.84	6.84	5	73.1% %
11	20.49	31.1	10.61	34.1%
82	72.26	94.7	22.44	23.7%

We immediately noticed the outlier of route 15. This brings us back to problems we dealt with collecting data for routes (see previous section). Since this is problematic and creates a signficant data outlier, we will proceed to remove route 15 from further consideration. While this is problematic, it clearly is not the longest route, thus it probably will not save us as much profit from the other routes anyway. Now we consider the profit from each route using the price of diesel as $2.915 per gallon [5]:

Route	Diesel cost with battery per day (USD)	Diesel cost without battery per day(USD)	Profit per day (USD)
81	17.65	31.60	13.95
10	101.18	158.69	57.51
11	59.73	174.11	114.38
82	210.64	276.05	65.41

From this, we can easily see that the TCAT should prioritize route 11 and route 82. This is a good choice for not only of economic priority, but also because these routes serve the most population: route 82 traverse through Central Campus to North Campus and route 11 serves the Ithaca College community. These two communities are those that are most populous and are central to the Ithaca community. Furthermore, these routes have a very small average slope (about 2%) and doesn't encounter many steep hills. The slope of Ithaca roads are presented below - figure 9 (purple to blue corresponds to less slope to more slope). We calculated this from our data and most of the slopes in Ithaca are about 3-7% and route 11 and 82 avoids most of the steep roads. Thus, it is safe to say that TCAT should prioritize routes 11 and 82 at all time. In addition, we recommend that the TCAT should serve 6 buses to route 11 and 2 to route 82 to capitalize on profit based on the profit difference between these two routes.

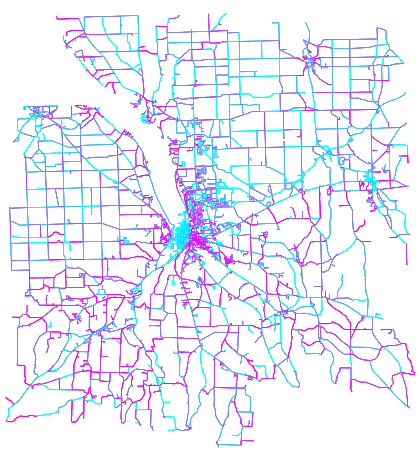

Figure 9: Slope of Ithaca - Purple corresponds to less slope, Blue corresponds to more slope

References

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7 Appendix

Our Python code is attached below. The library dependency for the code is:

- OSMnx
- Networkx
- Google Maps API Python Client